Robustness analysis through an information-based perspective

Daniel E. SALAZAR APONTE  
Computación Evolutiva y Aplicaciones – Universidad de Las Palmas de Gran Canaria  
danielsalazaraponte@gmail.com

Blas J. GALVÁN GONZÁLEZ  
Computación Evolutiva y Aplicaciones – Universidad de Las Palmas de Gran Canaria  
bgalvan@step.es

Claudio M. ROCCO SANSEVERINO  
Investigación de Operaciones y Computación – Universidad Central de Venezuela  
crocco@gmail.com
Contents

1. Motivation

2. Accounting for uncertainty

3. Robustness: a survey

4. Questions and comments

4. Robustness: Classes
1. Motivation
Motivation

- We are concerned about the use of metaheuristics to solve MCDM problems.
- Some questions:
  - What is the difference between problems with full information and problems with uncertainty?
  - What makes problem with uncertainty hard to solve?
  - Are the existing metaheuristics really useful to solve such problems?
Motivation

- We decided to investigate robustness to face uncertainty.

- Some questions:
  - What people think one says that something is robust?
  - Is there a universal concept of robustness? Is it possible to have it?
  - How robustness can be related or implemented with metaheuristics?
1. Motivation

2. Accounting for uncertainty in MCDM
Sources of uncertainty:
Types of uncertainty:

There is a long tradition in some disciplines like engineering and risk management of discriminating uncertainty in terms of its nature:

- **Type I:** also known as systemic or aleatory uncertainty, is distinguished by a variability that cannot be reduced by further empirical effort. In other words, such an uncertainty does not rely upon the information available but on the inherent randomness of the system under study and its environment.

- **Type II:** or epistemic uncertainty is the complement to the former. This kind of uncertainty is assumed to be reducible by additional information of the system or its environment. *Practitioners recognize the challenge of suitably represent epistemic uncertainty in opposition to aleatory uncertainty.*
Accounting for uncertainty in MCDM

\[ \kappa \rightarrow \Gamma(\kappa) \rightarrow \mathbf{K} \]

\[ \mathbf{x} \rightarrow \mathbf{X} \]

\[ F(\mathbf{x}) \rightarrow \mathbf{y} \rightarrow \mathbf{Y} \]
Accounting for uncertainty in MCDM

- Type I or aleatory uncertainty:
Accounting for uncertainty in MCDM

- Type I or aleatory uncertainty:
Accounting for uncertainty in MCDM

- Type II or epistemic uncertainty:
Accounting for uncertainty in MCDM

- Facts:
  - Both types of probabilities lead to the same problem: attributes are no longer represented by single crisp values but sets…
  - We do not know how to compare sets universally!!!

- Hypothesis:
  - Even when the consequence is the same in both cases, the causes are not (different spaces) so robustness does not have to be necessarily understood in the same way
Theoretical frameworks:

- Classical Probability Theory
- Fuzzy Logic
- Possibility Theory
- Imprecise probability theories:
  - Dempster-Shafer Theory
  - Walley’s imprecise probabilities
  - P-boxes
3. Robustness: A survey
Robustness: a survey

Problem 0:

Problem 1:

Problem 2:

These are PDF
Robustness: a survey

Problem 12:

These are fuzzy sets

Problem 13:

These are fuzzy sets
4. Robustness: From concepts to classes
Robustness: From concepts to classes

We define robustness in the more general way as:

\[ R(F(x), x, p, \delta_x, \delta_p, \gamma) \]

s.t.:

\[ x \in X, \ p \in P \]
\[ \delta_x \leq \delta_x \leq \bar{\delta}_x \]
\[ \delta_p \leq \delta_p \leq \bar{\delta}_p \]  \hspace{1cm} (4.12)

which can be read as the robustness of a system determined by its nominal decisional vector \( x \) in the decisional space \( X \), is a criterion defined in terms of the variation of its performance \( F(x) \), regarding a target quantity \( \gamma \), and the uncertainty associated with \( x \), namely \( \delta_x \), and its counterpart \( \delta_p \) in the environmental parameters’ space \( P \).

For simplicity, the uncertainty associated with \( x \) can be expressed as \( x + \delta_x \)
Definition 20 (Robustness-seeking program) let $F(x)$ be a measure of performance of a system determined by the decisional vector $x$ and influenced by a vector of environmental parameters $p$, each of which is subject of uncertainties $\delta_x$ and $\delta_p$ respectively. Let $G(\cdot)$ be a vector of constraints (inequality or equality) defined for the optimization of $F(x)$, and finally let $I(\cdot)$ be a vector of robustness constraints imposed upon the performance. The optimization of robustness consists in solving the following program

$$\text{Opt} \left( R(F(x), x, p, \delta_x, \delta_p, \gamma) \right)$$

s.t.:

$$x \in X, \ p \in P$$
$$\delta_x \leq \delta_x \leq \bar{\delta}_x$$
$$\delta_p \leq \delta_p \leq \bar{\delta}_p$$
$$G(x, p, \delta_x, \delta_p) \leq 0$$
$$I(F(x), x, p, \delta_x, \delta_p, \gamma) \leq 0$$

(4.13)
Class 1: Uncertainty propagating programs

This class, also known as *stochastic programming*, is characterized by a suitable description of $\underline{\delta}_x \leq \delta_x \leq \overline{\delta}_x$ and $\underline{\delta}_p \leq \delta_p \leq \overline{\delta}_p$ in such a way that the uncertainty can be propagated through $F(x)$. If the uncertainty is aleatory, $\delta_x$ and $\delta_p$ have associated PDF. For instance, $\delta_x$ could be a normally distributed number $N(0, \sigma)$ such that $\underline{\delta}_x = -\infty$ and $\overline{\delta}_x = \infty$. On the contrary, if the uncertainty is epistemic, $\underline{\delta}_x$ and $\overline{\delta}_x$ are finite scalars.
ROADEF 08:

ROBUSTNESS ANALYSIS THROUGH AN INFORMATION-BASED PERSPECTIVE

Daniel E. Salazar A., Blas J. Galván G. & Claudio M. Rocco S.

---

**Functions**

**Expected value as representative value:**

- \( R(F(x), x, \delta_x) = \frac{1}{n} \sum_{i=1}^{n} F(x + \delta_{x,i}) \)

**Robustness Analysis Through an Information-Based Perspective**

- \( R(F(x), x, \delta_x) = \left( f_1(x, \delta_x), \ldots, f_k(x, \delta_x) \right)^{t} \)

\[ s.t.: \quad I(F(x), x, \delta_x) = \frac{\left| f_i^{\text{eff}}(x + \delta_{x,i}) - f_i(x) \right|}{|f_i(x)|} \leq \eta, \]

where

\[ f_i^{\text{eff}}(x, \delta_x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x + \delta_{x,i}) \]

and

\( F(x) = (f_1(x), f_2(x), \ldots, f_k(x))^{t} \)

**Expected value and variance as representative values:**

- \( R(F(x), x, \delta_x) = \frac{\sigma[F(x)]}{\sigma[x]} \) where \( \sigma[] \) is the standard deviation of its argument and \( \sigma[x] \) is the average of \( \sigma[x_i] \).

- \( R(F(x), x, \delta_x) = w_1 E[F(x)] - w_2 \sigma^2[F(x)] \) where \( E[] \) is the expected value of its argument.

- \( R(F(x), x, \delta_x) = (E[F(x)], -\sigma^2[F(x)])^{t} \) where \( \sigma^2[:] \) is the variance of its argument.

**Extreme values as representative values:**

- \( R(F(x), x, \delta_x) = \text{worst case} \{ F(x + \delta_{x,i}) : \forall i \} \)

- \( R(F(x), x, \delta_p) = \text{worst case} \{ F(x + \delta_{p,i}) : \forall i \} \)

where the scenarios are defined as realizations of vector \( p \).

---

**Authors and works**


- K. Deb & H. Gupta [33].

- Y. Jin & B. Sendhoff [88].

- D.W. Coit, T. Jin & N. Wattanapongsakorn [29].


- Y-S. Ong, P.B. Nair & K.Y. Lum [141].

- P. Kouvelis & G. Yu (cf. [77]).
Robustness Class 2

Class 2: Robust domain-seeking programs: Some situations may keep the uncertainty associated with the input variables from being propagated through $F(x)$ and therefore to be handled as a Class 1 robustness problem. Whether the uncertainty is purely epistemic or mixed in nature, any assertion about $\delta_x \leq \delta_x \leq \bar{\delta}_x$ or $\delta_p \leq \delta_p \leq \bar{\delta}_p$ entails making some assumptions that could lead to inadequate or even artificial estimations of some representative parameters in the objective space with the consequent misclassification of the optimal solutions of a Class 1 $R(\cdot)$. Class 2 is then characterized by the existence of constraints of the type $I(F(x), x, p, \delta_x, \delta_p, \gamma) \leq 0$ that constitute desired performance levels of attainment (quality requirements), and a robustness function $R(\cdot)$ that aims at maximizing the range of variation of the input variables.
Robustness Class 2

Figure 4.15: Examples of different inner hyper-boxes for specified (left) and unspecified (right) centres ($G(\cdot)$ and $I(\cdot)$ constraints are represented by dotted lines).
Robustness Class 3

**Class 3 : Mixed robust-seeking procedure :** The two classes of robustness studied heretofore, derive from a partial knowledge of the elements constituting such definition. However, if the DM and the analyst are unable to characterize the input uncertainty nor the desired performance levels of attainment, or alluding eq. 2, if they cannot set $\delta_x$, $\delta_p$ and $I(F(x), x, \delta_x, \gamma)$, the actual definition of the robustness function $R(\cdot)$ is not possible. It is therefore mandatory to generate information to help the DM to make their minds about $\delta_x$, $\delta_p$ or about $I(\cdot)$, in such a way that the problem collapses into a Class 1 or Class 2 program.
Robustness problems:

| Input: $x + \delta_x, p + \delta_p$ | Output: $y \in \mathcal{Y}, |\mathcal{Y}| > 1$ |
|-------------------------------------|------------------------------------------|
| $\delta_x$ | $\delta_p$ | $I(\cdot)$ definable | $I(\cdot)$ undefinable | Comparison of sets $\equiv$ Class 1 |
| None | None | | | Class 1 |
| None | Definable | | | Class 1 |
| None | Undefinable | | | Class 2 |
| Definable | None | | | Class 1 |
| Definable | Definable | | | Class 1 |
| Definable | Undefinable | | | Class 2 |
| Undefinable | None | | | Class 2 |
| Undefinable | Definable | | | Class 2 |
| Undefinable | Undefinable | | | Class 2 |
Open problems:

- If \( R() \) admits many definitions, say \( R_1 \) and \( R_2 \) and we know that \( R_1(a) > R_1(b) \), under what conditions or for what possible definitions of \( R_1(a) > R_1(b) \iff R_2(a) > R_2(b) \) holds?

- What does robustness mean for other theoretical frameworks like fuzzy sets, possibility theory or the imprecise probabilities theories? Some Class 1 definitions are possible, what about Class 2?

- How do these questions affect metaheuristics?
Thank you!

Questions…
**Realm of study** | **Authors and works**  
---|---  
**Fuzzy logic:**  
- Estimation of the mean and variance.  
**Dempster-Shafer:**  
- Computing mean and variance.  
  - V. Kreinovich, G. Xiang & S. Ferson [107]  
**p-boxes:**  
- Estimation of the expected value.  
  - M. Bruns, C.J.J. Paredis & S. Ferson [22, 23].  
**Intervals:**  
- Fast computation of statistics for intervals and bounding.  
  - G. Xiang [209], S. Ferson, L. Ginzburg, V. Kreinovich & J. Lopez [44].  
**Monte Carlo simulation:**  
- Techniques for processing interval uncertainty in engineering.  
  - V. Kreinovich, J. Beck, C. Ferregut, A. Sanchez, G. Keller, M. Averill & S. A. Starks [106].

**Table 4.8:** Some contributions to the calculation of the mean and variance of fuzzy or imprecise probability quantities that can help towards the resolution of Class 1 $R(\cdot)$ functions.